Bit Error Rate(BER) Analysis of 4H-PAM Truncate-and-Forward(TF) Protocol:

BPSK at Relay and Maximum Ratio Combining at the destination:

February 8, 2008

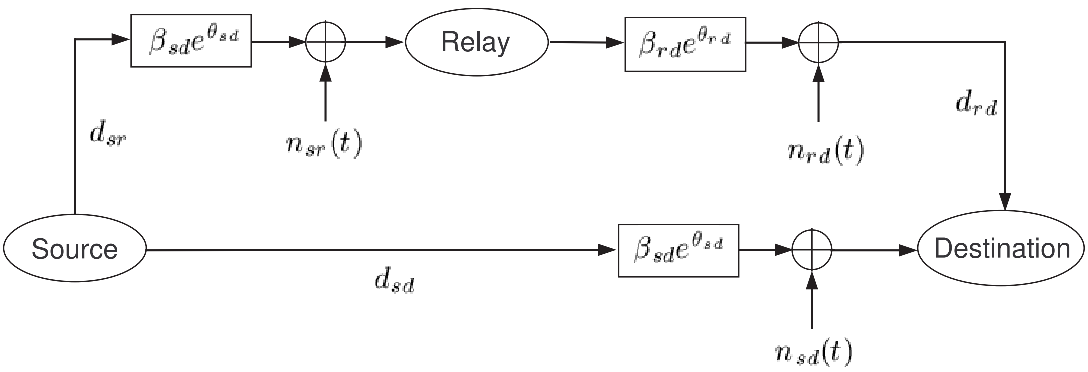

Figure 1: 4 H-PAM broadcasted by source(1st time slot) and Normalized BPSK sent by the relay(2nd time slot).

Above is the simplest form of cooperative system diagram

1 System Construction and MRC

After the source broadcasts 4H-PAM signal, let us now assume that it is preferable for the relay to send a BPSK signal instead of 4PAM in the second time slot, where each symbol has the same average energy compared to a 4H-PAM symbol. Therefore the symbol distance from the origin is $d^{'} = d\sqrt{\frac{(1+\alpha^2)}{2}}$ as shown in figure 1.

Now assume S-R channel is perfect and observe the following system diagram(At the end of 2^{nd} time slot):

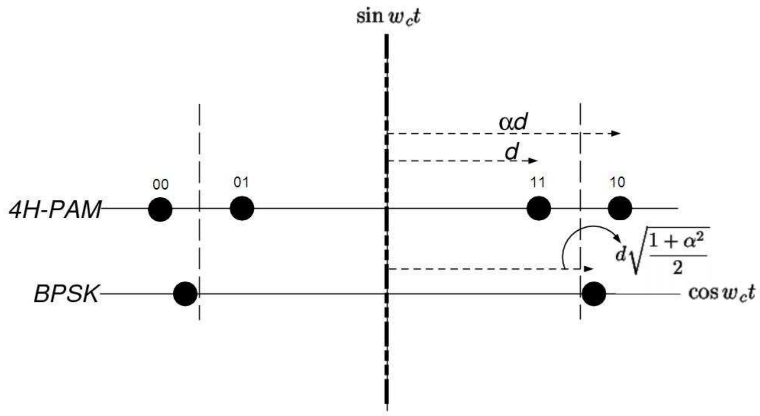

Figure 2: 4 H-PAM broadcasted by source(1st time slot) and Normalized BPSK sent by the relay(2nd time slot).

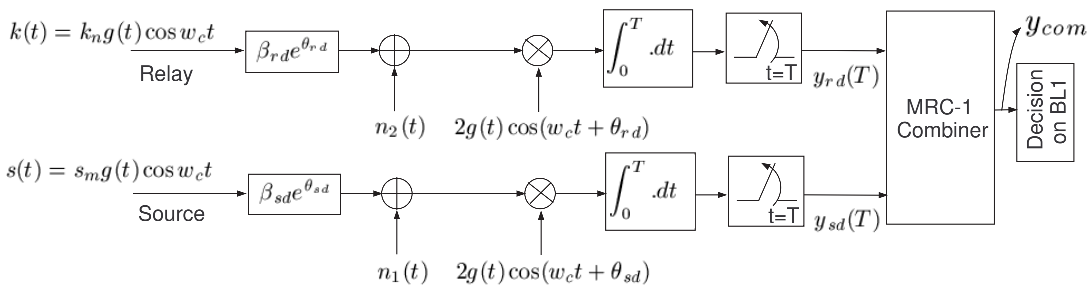

Figure 3: System Diagram when Relay transmits

Transmitted BPSK and 4H-PAM signals are given by,

$$k(t) = k_n g(t) \cos w_c t \tag{1}$$
$$s(t) = s_m g(t) \cos w_c t \tag{2}$$

where $k_n = \{\pm d\sqrt{\frac{(1+\alpha^2)}{2}}\}$,$s_m = \{\pm d, \pm\alpha d\}$ are the symbols, characterizing the distances from the origin, respectively. And α is the parameter that controls the hierarchy(See Figure 2). Also, $g(t)$ is the pulse shaping function. The subscripts are denoting the symbol indexes and takes on values $n = \{0,1\}$ for BPSK and $m = \{10, 11, 01, 00\}$ for 4-HPAM.

After match filtering(demodulation) before doing a hard decision, the destination uses MRC(Maximum ratio combining) to combine those symbols. The decision statistics before MRC can be expressed as:

$$y_{sd}(T) = \beta_{sd} s_m \varepsilon_g + n_{sd}(T) \tag{3}$$
$$y_{rd}(T) = \beta_{rd} k_n \varepsilon_g + n_{rd}(T) \tag{4}$$

where β_{sd}, β_{rd} are slowly varying flat rayleigh fading coefficients with parameters σ_{sd}, σ_{rd}, respectively. WOLOG let us drop T for simple illustration.

Therefore assuming static flat rayleigh fading, and weights μ_1 and μ_2, we have

$$\begin{aligned}
y_{sd} &= \beta_{sd} s_m \varepsilon_g + n_{sd} && (5)\\
y_{rd} &= \beta_{rd} k_n \varepsilon_g + n_{rd} && (6)\\
y_{com} &= \mu_{sd} y_{sd} + \mu_{rd} y_{rd} && (7)\\
&= \mu_{sd}\beta_{sd} s_m \varepsilon_g + \mu_{rd}\beta_{rd} k_n \varepsilon_g + \mu_{sd} n_{sd} + \mu_{rd} n_{rd} && (8)
\end{aligned}$$

where n_{sd}, n_{rd} are samples of a gaussian process with variance $\sigma^2_{n_{sd}}$ and $\sigma^2_{n_{rd}}$, respectively. It is found in the foregoing discussion that $\sigma^2_{n_{sd}} = N_0\varepsilon_g$, since the second goes through the same low pass filtering we have $\sigma^2_{n_{rd}} = N_0\varepsilon_g$. Then the output SNR can be written as follows and it is subject to an upper bound.

$$\textbf{Claim:} \qquad SNR^O_{output} = \frac{\frac{1}{2}E[\mu_{sd}\beta_{sd} s_m \varepsilon_g + \mu_{rd}\beta_{rd} k_n \varepsilon_g]^2}{var\{\mu_{sd} n_{sd} + \mu_{rd} n_{rd}\}} \underbrace{\leq}_{(\alpha\geq 1)} \frac{\varepsilon_{av}\varepsilon_g[\mu_{sd}\beta_{sd} + \mu_{rd}\beta_{rd}]^2}{(\mu^2_{sr} + \mu^2_{rd})\varepsilon_g N_0} \tag{9}$$

where ε_{av} is average symbol energy for 4H-PAM.

In a 4H-PAM signalling, the mean energy ε_{av} is given by

$$\begin{aligned}
\varepsilon_{av}^{4H-PAM} &= E\left[\int_0^T (s_m g(t) cos w_c t)^2\, dt\right] && (10)\\
&= E\left[\frac{1}{2}\int_0^T s_m^2 g^2(t) dt\right] = E\left[\frac{s_m^2}{2}\int_0^T g^2(t) dt\right] && (11)\\
&= \frac{\varepsilon_g}{2} E\left[s_m^2\right] && (12)\\
&= \frac{\varepsilon_g}{2}\frac{1}{4}\sum_{m=1}^{4} s_m^2 = \frac{\varepsilon_g (1+\alpha^2) d^2}{4}. && (13)
\end{aligned}$$

and mean energy for our normalized BPSK is also given by

$$\begin{aligned}
\varepsilon_{av}^{BPSK} &= E\left[\int_0^T \left(k_n g(t) cos w_c t\right)^2 dt\right] &(14)\\
&= E\left[\frac{1}{2}\int_0^T k_n^2 g^2(t)dt\right] = E\left[\frac{k_n^2}{2}\int_0^T g^2(t)dt\right] &(15)\\
&= \frac{\varepsilon_g}{2} E\left[k_n^2\right] &(16)\\
&= \frac{\varepsilon_g}{2}\frac{1}{2}\sum_{n=1}^{2} k_n^2 = \frac{\varepsilon_g(1+\alpha^2)d^2}{4}. &(17)
\end{aligned}$$

So they are equal as intended.

1.1 Probability of Error Expression

In this section we will first derive the BL bit error rate(BER) for three protocols:Truncate-and-Forward(TF), Decode-and-Forward(DF) and Amplify-and-Forward(AF), respectively. We will also sometimes resort to single link BL and EL BER performance expressions as needed. Then we will discuss EL BER expressions for these protocols and give finally numerical results and discussion.

1.2 Truncate-and-Forward

Since μ_1, μ_2 are chosen independent of α, the set $\left\{\mu_{sd} = c\frac{\beta_{sd}^*}{\varepsilon_g N_0}, \mu_{sd} = c\frac{\beta_{sd}^*}{\varepsilon_g N_0}, c \in \mathbb{N}\right\}$ that maximizes SNR_{output}^{DF} will also maximize the SNR_{output}^{O}. Without loss of generality, choose $c = N_0$, then we have:

$$y_{com} = \beta_{sd}^* y_{sd} + \beta_{rd}^* y_{rd} = |\beta_{sd}|^2 s_m \varepsilon_g + |\beta_{rd}|^2 k_n \varepsilon_g + \beta_{sd}^* n_{sd} + \beta_{rd}^* n_{rd}$$

where $s_m = \{\pm d, \pm\alpha d\}$ is the 4 H-PAM symbol sent by the source in the first time slot, and $k_n = \{\pm d\sqrt{\frac{1+\alpha^2}{2}}\}$ is the fixed normalized BPSK symbol sent by the relay and β_{sd}, β_{rd} are complex rayleigh fading coefficients with parameters σ_{sd}, σ_{rd} respectively. Therefore $|\beta_{sd}|^2$ and $|\beta_{rd}|^2$ are exponentially distributed with parameters $\frac{1}{2\sigma_{sd}^2}, \frac{1}{2\sigma_{rd}^2}$. Let us denote $|\beta_{sd}|^2, |\beta_{rd}|^2$ as x_{sd}, x_{rd} then the probability density functions are given by

$$f_{\underline{x_{sd}}}(x_{sd}) = \frac{1}{2\sigma_{sd}^2} e^{\frac{x_{sd}}{2\sigma_{sd}^2}} u(x_{sd}) \tag{18}$$

$$f_{\underline{x_{rd}}}(x_{rd}) = \frac{1}{2\sigma_{rd}^2} e^{\frac{x_{rd}}{2\sigma_{rd}^2}} u(x_{rd}) \tag{19}$$

where $u(.)$ is unit step function. Since x_{sd}, x_{rd}, s_m, k_n are all random, we first condition on them and then average over them. Assuming independent fading then we have the conditional probability of error expression that can be written as:

$$P(e/x_{sd}, x_{rd}, s_m, k_n) = P(\beta_{sd}^* n_{sd} + \beta_{rd}^* n_{rd} < -|\beta_{sd}|^2 s_m \varepsilon_g - |\beta_{rd}|^2 k_n)\varepsilon_g \tag{20}$$

$$P(e/x_{sd}, x_{rd}, s_m, k_n) = Q\left(\frac{x_{sd} s_m \varepsilon_g + x_{rd} k_n \varepsilon_g}{var\{\beta_{sd}^* n_{sd} + \beta_{rd}^* n_{rd}\}}\right) \tag{21}$$

$$= Q\left(\frac{x_{sd} s_m \varepsilon_g + x_{rd} k_n \varepsilon_g}{\sqrt{N_0 \varepsilon_g (x_{sd} + x_{rd})}}\right) \tag{22}$$

where $Q(.)$ function is given by

$$Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-\frac{z^2}{2}} dz. \tag{23}$$

As we have mentioned, we primarily assume a perfect channel between source and relay to facilitate the expressions. Note that s_m and k_n are then dependent and k_n takes values based on s_m.

$$k_n = \begin{cases} +\sqrt{\frac{1+\alpha^2}{2}} d & \text{if } s_m = +d, +\alpha d \\ -\sqrt{\frac{1+\alpha^2}{2}} d & \text{if } s_m = -d, -\alpha d \end{cases}$$

Therefore the average probability of error for BL bits can be expressed as:

$$P_+(e) = \frac{1}{4} \frac{1}{2\sigma_{s,d}^2} \frac{1}{2\sigma_{r,d}^2} \sum_{s_m} \sum_{k_n} \int_0^\infty \int_0^\infty Q\left(\frac{x_{sd} s_m \varepsilon_g + x_{rd} k_n \varepsilon_g}{\sqrt{N_0 \varepsilon_g (x_{sd} + x_{rd})}}\right) e^{-\frac{1}{2\sigma_{s,d}^2} x_{sd}} e^{-\frac{1}{2\sigma_{r,d}^2} x_{rd}} dx_{sd} dx_{rd} \tag{24}$$

$$= \frac{1}{2} \frac{1}{4\sigma_{s,d}^2 \sigma_{r,d}^2} \int_0^\infty \int_0^\infty Q\left(\frac{x_{sd} d \varepsilon_g + x_{rd} d \kappa \varepsilon_g}{\sqrt{N_0 \varepsilon_g (x_{sd} + x_{rd})}}\right) e^{-\frac{1}{2}\left(\frac{x_{sd}\sigma_{r,d}^2 + x_{rd}\sigma_{s,d}^2}{\sigma_{s,d}^2 \sigma_{r,d}^2}\right)} dx_{sd} dx_{rd} \tag{25}$$

$$+\frac{1}{2} \frac{1}{4\sigma_{s,d}^2 \sigma_{r,d}^2} \int_0^\infty \int_0^\infty Q\left(\frac{x_{sd} \alpha d \varepsilon_g + x_{rd} d \kappa \varepsilon_g}{\sqrt{N_0 \varepsilon_g (x_{sd} + x_{rd})}}\right) e^{-\frac{1}{2}\left(\frac{x_{sd}\sigma_{r,d}^2 + x_{rd}\sigma_{s,d}^2}{\sigma_{s,d}^2 \sigma_{r,d}^2}\right)} dx_{sd} dx_{rd} \tag{26}$$

where $\kappa = \sqrt{\frac{(1+\alpha^2)}{2}}$. I used numerical integration tools to evaluate that.

Now let us assume $\varepsilon_g = 1$ and define the following function to ease the later expressions:

$$\psi(A, B, C, D) = \frac{\int_0^\infty \int_0^\infty \frac{1}{2}\left[Q\left(\frac{x_{sd}A + x_{rd}B}{\sqrt{N_0(x_{sd}+x_{rd})}}\right) + Q\left(\frac{x_{sd}C + x_{rd}D}{\sqrt{N_0(x_{sd}+x_{rd})}}\right)\right] e^{-\frac{1}{2}\left(\frac{x_{sd}\sigma_{r,d}^2 + x_{rd}\sigma_{s,d}^2}{\sigma_{s,d}^2 \sigma_{r,d}^2}\right)} dx_{sd} dx_{rd}}{4\sigma_{s,d}^2 \sigma_{r,d}^2}$$

Therefore $P_+(e) = \psi(d, d\kappa, \alpha d, d\kappa)$. This expression can also be written in terms of ε_{av}[1] as well:

$$P_+(e) = \psi\left(\frac{2\sqrt{\varepsilon_{av}}}{\sqrt{1+\alpha^2}}, \sqrt{2\varepsilon_{av}}, \frac{2\alpha\sqrt{\varepsilon_{av}}}{\sqrt{1+\alpha^2}}, \sqrt{2\varepsilon_{av}}\right). \tag{27}$$

[1] Note that In this section we refer $\varepsilon_{av}^{4H-PAM}$ as ε_{av} for simplicity.

Now for the next step, let us assume S-R link is not perfect and subject to information loss. This is a simple two terminal single rayleigh fading link. We found a probability of error expression for this in the first part as follows:

$$P_e^{bl} = \frac{1}{2} - \frac{1}{4}\sqrt{\frac{\lambda}{1+\lambda}} - \frac{1}{4}\sqrt{\frac{\alpha^2\lambda}{1+\alpha^2\lambda}} \tag{28}$$

where $\lambda = \frac{\sigma_{s,r}^2 d^2}{N_0} = \frac{4\sigma_{s,r}^2\varepsilon_{av}}{(1+\alpha^2)N_0}$, and $\sigma_{s,r}$ is the parameter of the rayleigh fading coefficient $\beta_{s,r}$. If relay makes error in BL bits, then in this case the probability of error expression will be:

$$\begin{aligned} P_-(e) &= \frac{1}{2}\frac{1}{4\sigma_{s,d}^2\sigma_{r,d}^2}\int_0^\infty\int_0^\infty \left[Q\left(\frac{x_{sd}d - x_{rd}d\kappa}{\sqrt{N_0(x_{sd}+x_{rd})}}\right) + Q\left(\frac{x_{sd}\alpha d - x_{rd}d\kappa}{\sqrt{N_0(x_{sd}+x_{rd})}}\right)\right] e^{-\frac{1}{2}\left(\frac{x_{sd}\sigma_{r,d}^2 + x_{rd}\sigma_{s,d}^2}{\sigma_{s,d}^2\sigma_{r,d}^2}\right)} dx_{sd}dx_{rd} \\ &= \psi(d, -d\kappa, \alpha d, -d\kappa) \\ &= \psi(\frac{2\sqrt{\varepsilon_{av}}}{\sqrt{1+\alpha^2}}, -\sqrt{2\varepsilon_{av}}, \frac{2\alpha\sqrt{\varepsilon_{av}}}{\sqrt{1+\alpha^2}}, -\sqrt{2\varepsilon_{av}}). \end{aligned} \tag{29}$$

Then for any relay position, the probability of error expression for TF scheme can be written in the following form:

$$P^{TF}(e) = (1-P_e^{bl})P_+(e) + P_e^{bl}P_-(e). \tag{30}$$

with $\sigma_{s,d}^2 = 1$ and $\sigma_{s,r}^2 = \frac{1}{d_{s,r}^n}$,$\sigma_{r,d}^2 = \frac{1}{d_{r,d}^n}$ where n is the path loss exponent, and $d_{s,r}, d_{r,d}$ are the corresponding distances between terminals.

Note that each expression is expressed in terms of ε_{av} so let us write BER in the following way for future reference:

$$P_{e,b}^{TF}(\varepsilon_{av}) = (1-P_e^{bl})P_+(e) + P_e^{bl}P_-(e). \tag{31}$$

1.3 Decode-and-Forward

Let us consider now the probability of error expression of BL bits for DF protocol, the expressions are very similar except now the relay sends another 4H-PAM. Therefore unlike TF, the symbol sent by the relay depends on the EL bits. So the error probability of EL between S-R will ultimately effect the overall BL BER.

BL BER is given for a point-to-point(*ptp*) comm. under slow rayleigh flat fading channel and denoted as P_e^{bl}. For EL, *ptp* BER can be found in a closed form and given by after some modification of the form found in previous discussion:

$$P_e^{el} = \frac{1}{2} + \frac{1}{2}\sum_{s=0}^{1}\sum_{k=0}^{1}(-1)^{ks}\left(\frac{1}{2} - \frac{\mathrm{sgn}(L)}{2\sqrt{1+\frac{N_0}{L^2\sigma_{s,r}^2}}}\right). \tag{32}$$

where $L = (-1)^s W + \alpha^k d$, $W = \frac{1+\alpha}{2}d$ and sgn(.) is the signum function and is given by

$$\mathrm{sgn}(x) = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \\ 0 & \text{otherwise.} \end{cases}$$

Note that in order to find exact expression for any relay position we need to consider for two symbols say s_{11} and s_{10} each symbol error probability which amounts to calculate 8 different cases. Because of symmetry we can average over only these two symbols.

Instead of using above method, we can find an expression using already existing BER expressions. Therefore using single link BL and EL BER expressions, we can write the probability of error expression for BL bits of our DF protocol in the following way:

$$\begin{aligned} P^{DF}(e) &\simeq (1-P_e^{bl})(1-P_e^{el})\psi(\alpha d, \alpha d, d, d) && (33)\\ &+ (1-P_e^{bl})P_e^{el}\psi(\alpha d, d, d, \alpha d) && (34)\\ &+ P_e^{bl}(1-P_e^{el})\psi(\alpha d, -\alpha d, d, -d) && (35)\\ &+ P_e^{bl}P_e^{el}\psi(\alpha d, -d, d, -\alpha d). && (36)\end{aligned}$$

1.3.1 Probability of Error Expression for EL

Now assume S-R is perfect, and remember that the combined signal of our system is given by

$$y_{com} = \beta_{sd}^* y_{sd} + \beta_{rd}^* y_{rd} = |\beta_{sd}|^2 s_m + |\beta_{rd}|^2 s_m + \beta_{sd}^* n_{sd} + \beta_{rd}^* n_{rd}$$

Again, since the optimal decision regions for EL is not simply the origin, we need to compensate for the fading before finding the test statistics and probability of error expression. The compensated signal will be

$$\begin{aligned} y_{com} &= \frac{|\beta_{sd}|^2 s_m + |\beta_{rd}|^2 s_m}{|\beta_{sd}|^2 + |\beta_{rd}|^2} + \frac{\beta_{sd}^* n_{sd} + \beta_{rd}^* n_{rd}}{|\beta_{sd}|^2 + |\beta_{rd}|^2} && (37)\\ &= S(T) + N(T) \end{aligned}$$

If we assume signal is never in error when received at relay then we have:

$$\begin{aligned} S(T) &= s_m && (38)\\ N(T) &\sim \mathcal{N}\left(0, \frac{N_0 \varepsilon_g}{|\beta_{sd}|^2 + |\beta_{rd}|^2}\right) && (39)\end{aligned}$$

let us denote $|\beta_{sd}|^2 \sim x_{sd}$ and $|\beta_{rd}|^2 \sim x_{rd}$ are exponential distributed r.v.'s as defined before. Considering s_{11}

and s_{10} will suffice, because of symmetry. Probability of Error conditioned on channel parameters is given by

$$\begin{aligned}
P_{el}^{DF}(e/x_{sd},x_{rd}) &= \frac{1}{2}(1-P_c(s_{11}))+\frac{1}{2}P_e(s_{10}) && (40)\\
&= \frac{1}{2}-\frac{1}{2}P(-W<s_{11}+N(T)<W)+\frac{1}{2}P(-W<s_{10}+N(T)<W) && (41)\\
&= \frac{1}{2}-\frac{1}{2}P(-W-s_{11}<N(T)<W-s_{11})+\frac{1}{2}P(-W-s_{10}<N(T)<W-s_{10}) && (42)\\
&= \frac{1}{2}+\frac{1}{2}\left(Q\left((-W+\alpha d)\sqrt{\frac{x_{sd}+x_{rd}}{N_0\varepsilon_g}}\right)-Q\left((W+\alpha d)\sqrt{\frac{x_{sd}+x_{rd}}{N_0\varepsilon_g}}\right)\right) && (43)\\
&\quad +\frac{1}{2}\left(Q\left((W+d)\sqrt{\frac{x_{sd}+x_{rd}}{N_0\varepsilon_g}}\right)-Q\left((-W+d)\sqrt{\frac{x_{sd}+x_{rd}}{N_0\varepsilon_g}}\right)\right) && (44)\\
&= \frac{1}{2}+\frac{1}{2}Q\left(\frac{x_{sd}(-W+\alpha d)+x_{rd}(-W+\alpha d)}{\sqrt{(x_{sd}+x_{rd})N_0\varepsilon_g}}\right)+\frac{1}{2}Q\left(\frac{x_{sd}(W+d)+x_{rd}(W+d)}{\sqrt{(x_{sd}+x_{rd})N_0\varepsilon_g}}\right)\\
&\quad -\frac{1}{2}Q\left(\frac{x_{sd}(W+\alpha d)+x_{rd}(W+\alpha d)}{\sqrt{(x_{sd}+x_{rd})N_0\varepsilon_g}}\right)-\frac{1}{2}Q\left(\frac{x_{sd}(-W+d)+x_{rd}(-W+d)}{\sqrt{(x_{sd}+x_{rd})N_0\varepsilon_g}}\right) && (45)\\
&= \frac{1}{2}+\frac{1}{2}\sum_{s=0}^{1}\sum_{k=0}^{1}(-1)^{sk}Q\left(((-1)^sW+\alpha^k d)\sqrt{\frac{x_{sd}+x_{rd}}{N_0\varepsilon_g}}\right) && (46)
\end{aligned}$$

Therefore the unconditional probability of error can be given as

$$P_{e,el}^{DF}=\frac{1}{4\sigma_{sd}^2\sigma_{rd}^2}\int_0^\infty\int_0^\infty P_{el}^{DF}(e/x_{sd},x_{rd})\exp\left\{-\frac{1}{2}\left(\frac{x_{sd}\sigma_{r,d}^2+x_{rd}\sigma_{s,d}^2}{\sigma_{s,d}^2\sigma_{r,d}^2}\right)\right\}dx_{sd}dx_{rd} \quad (47)$$

Note that this expression can be written in terms of $\psi(A,B,C,D)$ function we defined. Using the form in (45) we can express it as

$$P_{e,el}^{DF}=0.5+\psi(-W+\alpha d,-W+\alpha d,W+d,W+d)-\psi(W+\alpha d,W+\alpha d,-W+d,-W+d). \quad (48)$$

where $W=\frac{1+\alpha}{2}d$ is the place on x axis where the decision region resides.

If S-R link is not perfect the conditional probability of expression will be given by

$$\begin{aligned}
P_{e,el}^{DF} &= (1-P_e^{bl})(1-P_e^{el})\Big(0.5+\psi(-W+\alpha d,-W+\alpha d,W+d,W+d)-\psi(W+\alpha d,W+\alpha d,-W+d,-W+d)\Big)\\
&\quad +(1-P_e^{bl})P_e^{el}\Big(0.5+\psi(-W+\alpha d,-W+d,W+d,W+\alpha d)-\psi(W+\alpha d,W+d,-W+d,-W+\alpha d)\Big) && (49)\\
&\quad +P_e^{bl}(1-P_e^{el})\Big(0.5+\psi(-W+\alpha d,-W-\alpha d,W+d,W-d)-\psi(W+\alpha d,W-\alpha d,-W+d,-W-d)\Big) && (50)\\
&\quad +P_e^{bl}P_e^{el}\Big(0.5+\psi(-W+\alpha d,-W-d,W+d,W-\alpha d)-\psi(W+\alpha d,W-d,-W+d,-W-\alpha d)\Big) && (51)
\end{aligned}$$

1.4 Amplify-and-Forward

Let us first consider the received signal at relay in the first time slot(T). Based on our previous discussion and construction the test statistic at relay can be found as:

$$y_{sr}(T) = \beta_{sr} s_m \varepsilon_g + n_{sr}(T) \tag{52}$$

Assuming perfect channel information, the power transmitted at the source is given by

$$P_s = \frac{\frac{1}{2}\varepsilon_g E[s_m^2]}{T} \tag{53}$$

Now let us assume that relay is subject to a power constraint, P_r. This immediately yields a bounded amplification factor ρ. In order to have relay transmit at the same power level, we multiply the received symbol with ρ such that:

$$\begin{aligned} P_r = \frac{\frac{1}{2}E[|\rho y_{sr}|^2]}{T} &\leq P_s = \frac{\frac{1}{2}\varepsilon_g E[s_m^2]}{T} \\ E[|\rho y_{sr}|^2] &\leq E[s_m^2]\varepsilon_g \end{aligned} \tag{54}$$

then we can now bound ρ and is given by

$$\rho^2(|\beta_{sr}|^2 E[s_m^2]\varepsilon_g^2 + N_0\varepsilon_g) \leq E[s_m^2]\varepsilon_g \tag{55}$$

$$\rho^2 \leq \frac{E[s_m^2]\varepsilon_g}{|\beta_{sr}|^2 E[s_m^2]\varepsilon_g^2 + N_0\varepsilon_g} \tag{56}$$

$$\rho^2 \leq \frac{\mathcal{SNR}}{|\beta_{sr}|^2 \mathcal{SNR}\varepsilon_g + 1} \tag{57}$$

$$\rho \leq \sqrt{\frac{\mathcal{SNR}}{|\beta_{sr}|^2 \mathcal{SNR}\varepsilon_g + 1}} \tag{58}$$

In the next time slot of duration T, while source stays idle relay forwards ρy_{sr} and destination uses MRC to combine signals coming from relay and source. Test statistics received at destination in both time slots are given by

$$y_{sd}(T) = \beta_{sd} s_m \varepsilon_g + n_{sd}(T) \tag{59}$$

$$y_{rd}(T) = \beta_{rd}\rho y_{sr}(T) + n_{rd}(T) \tag{60}$$

$$= \beta_{rd}\rho(\beta_{sr} s_m \varepsilon_g + n_{sr}(T)) + n_{rd}(T) \tag{61}$$

$$= \beta_{rd}\rho\beta_{sr} s_m \varepsilon_g + \beta_{rd}\rho n_{sr}(T) + n_{rd}(T) \tag{62}$$

Assuming perfect channel info available at the destination, MRC achieves the optimal performance by multiplying each received signal by its corresponding combining coefficients. Therefore the combined signal is given by

$$y_{com} = \mu_{sd} y_{sd} + \mu_{rd} y_{rd} \tag{63}$$

the coefficients can be shown to be,

$$\mu_{sd} = \beta_{sd}^* \tag{64}$$

$$\mu_{sd} = \frac{\beta_{rd}^* \rho^* \beta_{sr}^*}{|\beta_{rd}|^2 \rho^2 + 1} \tag{65}$$

The combined signal can be written as,

$$y_{com} = |\beta_{sd}|^2 s_m \varepsilon_g + \beta_{sd}^* n_{sd} + \frac{|\beta_{rd}|^2|\rho|^2|\beta_{sr}|^2}{|\beta_{rd}|^2\rho^2+1} s_m\varepsilon_g + \frac{|\beta_{rd}|^2|\rho|^2\beta_{sr}^*}{|\beta_{rd}|^2\rho^2+1} n_{sr} + \frac{\beta_{rd}^*\rho^*\beta_{sr}^*}{|\beta_{rd}|^2\rho^2+1} n_{sr} \quad (66)$$

$$= S(T) + N(T). \quad (67)$$

where the noise statistics conditioned on channel parameters are gaussian distributed and can be shown to be

$$N(T) \sim \mathcal{N}\left(0, \left(\frac{|\beta_{rd}|^2|\rho|^2|\beta_{sr}|^2}{|\beta_{rd}|^2|\rho|^2+1} + |\beta_{sd}|^2\right) N_0\varepsilon_g\right) \quad (68)$$

and the signal component is given by

$$S(T) = \left(\frac{|\beta_{rd}|^2|\rho|^2|\beta_{sr}|^2}{|\beta_{rd}|^2|\rho|^2+1} + |\beta_{sd}|^2\right) s_m\varepsilon_g \quad (69)$$

Let again $x_{sr} \sim |\beta_{sr}|^2$,$x_{sd} \sim |\beta_{sd}|^2$,$x_{rd} \sim |\beta_{rd}|^2$ be exponentially distributed random variables with parameters $\frac{1}{2\sigma_{sr}^2}$, $\frac{1}{2\sigma_{sd}^2}$, $\frac{1}{2\sigma_{rd}^2}$, respectively and define the following function:

$$\digamma(x_{sr}, x_{sd}, x_{rd}) = \frac{x_{rd}|\rho|^2 x_{sr}}{x_{rd}|\rho|^2+1} + x_{sd} \quad (70)$$

In order to have maximum amplification, let us substitute $\rho = \sqrt{\frac{\mathcal{SNR}}{x_{sr}\mathcal{SNR}\varepsilon_g+1}}$ then,

$$\digamma(x_{sr}, x_{sd}, x_{rd}) = \frac{x_{rd}x_{sr}\mathcal{SNR}}{(x_{rd}+x_{sr}\varepsilon_g)\mathcal{SNR}+1} + x_{sd} \quad (71)$$

The conditional probability of error conditioned on channel parameters is given by

$$P^{AF}(e/x_{sr}, x_{sd}, x_{rd}) = \frac{1}{2}P(S(T)+N(T)<0|s_m=d) + \frac{1}{2}P(S(T)+N(T)<0|s_m=\alpha d) \quad (72)$$

$$= \frac{1}{2}P(N(T)<-S(T)|s_m=d) + \frac{1}{2}P(N(T)<-S(T)|s_m=\alpha d) \quad (73)$$

$$= \frac{1}{2}Q\left(\frac{\digamma(x_{sr}, x_{sd}, x_{rd})d\varepsilon_g}{\sqrt{\digamma(x_{sr}, x_{sd}, x_{rd})N_0\varepsilon_g}}\right) + \frac{1}{2}Q\left(\frac{\digamma(x_{sr}, x_{sd}, x_{rd})\alpha d\varepsilon_g}{\sqrt{\digamma(x_{sr}, x_{sd}, x_{rd})N_0\varepsilon_g}}\right) \quad (74)$$

$$= \frac{1}{2}\left[Q\left(d\sqrt{\frac{\digamma(x_{sr}, x_{sd}, x_{rd})\varepsilon_g}{N_0}}\right) + Q\left(\alpha d\sqrt{\frac{\digamma(x_{sr}, x_{sd}, x_{rd})\varepsilon_g}{N_0}}\right)\right] \quad (75)$$

The probability of error expression for BL bits of AF protocol can be given by averaging this conditional probability of error expression over channel fading parameters:

$$P_b^{AF}(e) = \frac{1}{2}\frac{1}{8\sigma_{sr}^2\sigma_{sd}^2\sigma_{rd}^2}\int_0^\infty\int_0^\infty\int_0^\infty \left[Q\left(d\sqrt{\frac{\digamma(x_{sr}, x_{sd}, x_{rd})\varepsilon_g}{N_0}}\right) + Q\left(\alpha d\sqrt{\frac{\digamma(x_{sr}, x_{sd}, x_{rd})\varepsilon_g}{N_0}}\right)\right] \quad (76)$$

$$e^{-\frac{1}{2\sigma_{sr}^2}x_{sr}} e^{-\frac{1}{2\sigma_{sd}^2}x_{sd}} e^{-\frac{1}{2\sigma_{rd}^2}x_{rd}} dx_{sr}dx_{sd}dx_{rd} \quad (77)$$

2 Analysis extended to 16H-QAM

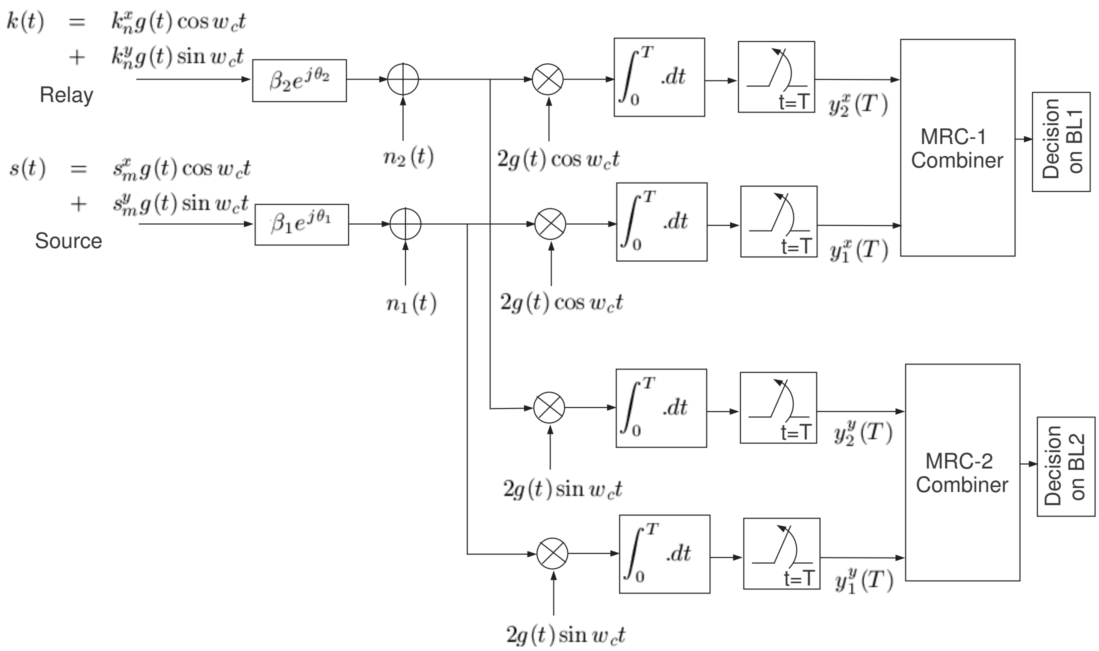

Figure 4: System Diagram of 16H-QAM when Relay transmits 4QAM

The system block diagram is show above. The relay transmits the following signal when it uses TF protocol:

$$k(t) = k_n^x g(t) \cos w_c t + k_n^y g(t) \sin w_c t \tag{78}$$

where

$$k_n^x = \pm d_i \sqrt{\frac{1+\alpha_i^2}{2}}, \quad k_n^y = \pm d_q \sqrt{\frac{1+\alpha_q^2}{2}} \tag{79}$$

Transmitted signal sent by the source is given by

$$s(t) = s_m^x g(t) \cos w_c t + s_m^y g(t) \sin w_c t \tag{80}$$

where

$$s_m^x = \pm d_i, \pm \alpha_i d_i \quad s_m^y = \pm d_q, \pm \alpha_q d_q \tag{81}$$

with $n_i(t), i = 1, 2$ is a zero-mean complex additive white gaussuian noise process sample with two sided spectral density of $N_0/2$. At the destination, the decision statistics can be given as:

$$\begin{aligned}
y_1^x(T) &= \tilde{\beta}_1 s_m^x \varepsilon_g + n_1(T) \\
y_2^x(T) &= \tilde{\beta}_2 k_n^x \varepsilon_g + n_2(T) \\
y_1^y(T) &= \tilde{\beta}_1 s_m^y \varepsilon_g + n_1(T) \\
y_2^x(T) &= \tilde{\beta}_2 k_n^y \varepsilon_g + n_2(T)
\end{aligned}$$

where $\widetilde{\beta}_i = \beta_i e^{\theta_i}, i = 1, 2$.Further statistics of noise is found already to be $n_i \sim \mathcal{N}(0, N_0\varepsilon_g)$. Similarly as before, we drop *tilde* for simplicity and treat β_i as a complex random variable with rayleigh amplitude and uniform phase. We also can drop T and assume $\varepsilon_g = 1$.

$$\varepsilon_{av}^{16H-QAM} = E\left[\int_0^T \left(s_m^x g(t)\cos w_c t + s_m^y g(t)\sin w_c t\right)^2 dt\right] \tag{82}$$

$$= \frac{\varepsilon_g}{2}\left(E\left[(s_m^x)^2\right] + E\left[(s_m^y)^2\right]\right) \tag{83}$$

$$= \frac{\varepsilon_g(1+\alpha_i^2)d_i^2}{4} + \frac{\varepsilon_g(1+\alpha_q^2)d_q^2}{4} = \frac{\varepsilon_g(1+\alpha^2)d^2}{2} = 2\varepsilon_{av}^{4H-PAM}. \tag{84}$$

where we used the following constraint $(1+\alpha_i^2)d_i^2 = (1+\alpha_q^2)d_q^2 = (1+\alpha^2)d^2$. We can also show,

$$\varepsilon_{av}^{4QAM} = \frac{\varepsilon_g(1+\alpha^2)d^2}{2} = 2\varepsilon_{av}^{4H-PAM}. \tag{85}$$

The noise samples $n_1(t), n_2(t)$ are i.i.d. gaussian process samples. The BER expression for BL in our cooperative system can be determined from that of 4H-PAM signalling with $\varepsilon_{av}^{4H-PAM} \longleftrightarrow \frac{\varepsilon_{av}^{16H-QAM}}{2}$.

Using (31), we will be able to write the SER expression for 16H-QAM system. Probability of correct decision is given by

$$P_{s,correct}^{TF,16H-QAM} = \left(1 - P_b^{TF}\left(\frac{\varepsilon_{av}^{16H-QAM}}{2}\right)\right)^2 \tag{86}$$

Therefore the symbol error rate can be written as:

$$P_s^{TF,16H-QAM} = 1 - \left(1 - P_b^{TF}\left(\frac{\varepsilon_{av}^{16H-QAM}}{2}\right)\right)^2. \tag{87}$$