On Hard Decision Upper Bounds for Coded M-Ary Hierarchical Modulation

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Introduction

- Hierarchical modulation is used in conjunction with error correction coding in many multimedia communication systems to provide unequal error protection.
- Hard decision decoding may be well suited for receivers that require low complexity implementation.
- A receiver might be equipped with both hard decision and soft decision decoding capability for possible savings on energy consumption.
- Convolutional codes are best known for their trellis representation and efficient decoding (Viterbi decoding).
- Block codes are shown to be represented by efficient trellis structures.

Hierarchical Modulations: UEP

- α_i is the hierarchical parameter. Hierarchical parameter is used to adjust distances in the constellation. HP: High Priority, LP: Low Priority. HP and LP bits have different average BERs even if $\alpha_i = 3$ i.e., conventional modulation.

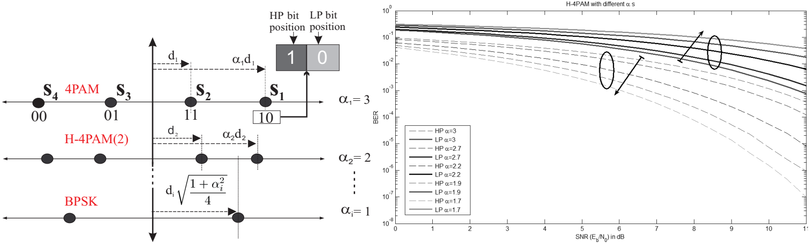


Figure: UEP by Hierarchical-4PAM (H-4PAM). Average symbol energy is kept constant.

Convolutional Codes & Block codes

- Convolutional codes are the most popular. An actual encoded sequence can be represented as a path in a trellis. These codes are often implemented in concatenation with a block code.
- In [1], the trellis representation of a linear block code is considered. Viterbi algorithm is used for maximum likelihood (ML) decoding. Prohibitively complex.
- Later, efficient and less complex approaches are proposed for trellis decoding of block codes.

[1] L. R. Bahl, J. Cocke, F. Jelinek, and J. Raviv, "Optimal decoding of Linear Codes for minimizing symbol error rate," IEEE Trans. Inform. Theory, vol. IT-20, pp. 284-287, 1974

Previous studies and the problem statement

- Previous studies for coded hierarchical modulated systems are too specific to the topology of the constellations and consider only a narrow class of codes such as block codes [1].
- The expressions are also specific to the channel models used.
- A more unified and an efficient hard decision bound is derived in this study for a given set of parameters of the channel model and the hierarchical constellation topology.

[2] P.K. Vitthaladevuni and M.S. Alouini, "An upper bound on the BER of block coded hierarchical constellations," 2003 IEEE Pacific Rim Conference on. Communications, Computers and Signal Processing (PACRIM 03), vol. 2, pp. 950-953. Aug. 2003.

The system model

- Consider the following coded system model with 2-layer transmission:

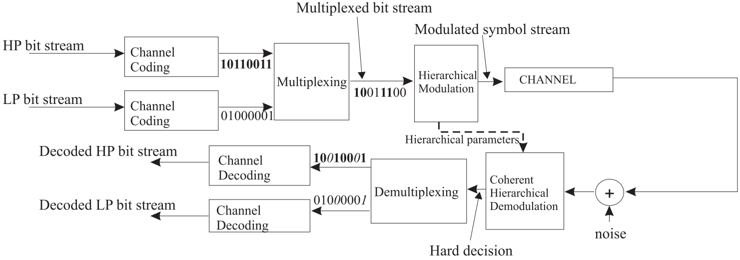


Figure: System block diagram for 2-layer transmission. HP: High Priority, LP: Low Priority.

An Observation about a coded hierarchically modulated system

- H-4PAM constellation consists of $\{s_1, s_2, s_3, s_4\}$. Define error event e_{ij} when bit $i \in \{0,1\}$ flips to bit $j \in \{0,1\}, i \neq j$.
- Consider the HP bit. Assume a memoryless channel such as AWGN or flat Rayleigh channel. Let $\epsilon_1 \triangleq P(e_{10}|s_1 \text{is transmitted})$ and $\epsilon_2 \triangleq P(e_{10}|s_2 \text{is transmitted})$ where

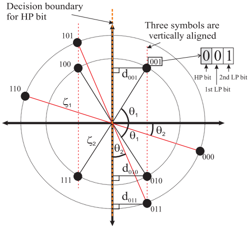
$$(\epsilon_1, \epsilon_2) = \begin{cases} \text{AWGN:} & \left(Q\left(\sqrt{\frac{8\gamma}{1+\alpha^2}}\right), Q\left(\sqrt{\frac{8\gamma\alpha^2}{1+\alpha^2}}\right)\right) \\ \text{Flat Rayleigh:} & \left(\frac{1}{2} - \frac{1}{2}\sqrt{\frac{\lambda}{1+\lambda}}, \frac{1}{2} - \frac{1}{2}\sqrt{\frac{\alpha^2\lambda}{1+\alpha^2\lambda}}\right) \end{cases}$$

where $\lambda = \frac{8\sigma^2\gamma}{1+\alpha^2}$, $\gamma (= \frac{E_b}{N_0})$ is the average SNR per bit, α is the hierarchical parameter, σ is the parameter of the Rayleigh distribution and $Q(z) = \frac{1}{\sqrt{2\pi}} \int_z^\infty e^{-x^2/2} dx$.

- Similar set of error probabilities can be generate for LP bits.
- In a given hierarchical constellation, suppose that class i contains the symbols that correspond to an error probability of ϵ_i. We define p_i to be the probability of randomly drawing any symbol from class i.
- **Observation:** Depending on which symbol in the constellation is transmitted, the bits in any of the priority classes experience different channel BERs.

Parameter values for hierarchical constellations

- Note that $p_1 = 0.5$ for H-4PAM, because selection of LP bits are equally likely. There are constellations with $p \neq 0.5$ as shown in the Figure (Concentric 2/4PSK contellations).
- Parameters for both H-4PAM and Concentric 2/4PSK is given in the Table below.
- Note that for these examples since $p_1 + p_2 = 1$, we only show p_1 in the Table.



Two concentric hierarchical 2/4 PSK [10]

Figure: A hierarchical constellation where we have $p_1 = 0.75$.

Modulation	Priority	Channel	ϵ_1	ϵ_2	p_1
H-4PAM/H-16QAM	HP	AWGN	$Q(\sqrt{\frac{8\gamma}{1+\alpha^2}})$	$Q(\sqrt{\frac{8\gamma\alpha^2}{1+\alpha^2}})$	0.5
H-4PAM/H-16QAM	HP	Rayleigh	$\frac{1}{2} - \frac{1}{2}\sqrt{\frac{\lambda}{1+\lambda}}$	$\frac{1}{2} - \frac{1}{2}\sqrt{\frac{\alpha^2\lambda}{1+\alpha^2\lambda}}$	0.5
Concentric H-2/4PSK	HP	AWGN	$Q(\varsigma_1 \cos\theta_1 \sqrt{3\gamma})$ [?]	$Q(\varsigma_2 \cos\theta_2 \sqrt{3\gamma})$ [?]	0.75

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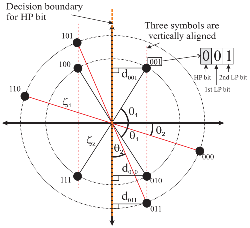

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H-4PAM/H-16QAM	HP	Rayleigh	$\frac{1}{2} - \frac{1}{2}\sqrt{\frac{\lambda}{1+\lambda}}$	$\frac{1}{2} - \frac{1}{2}\sqrt{\frac{\alpha^2\lambda}{1+\alpha^2\lambda}}$	0.5
Concentric H-2/4PSK	HP	AWGN	$Q(\varsigma_1 \cos\theta_1 \sqrt{3\gamma})$ [?]	$Q(\varsigma_2 \cos\theta_2 \sqrt{3\gamma})$ [?]	0.75

The main result

- A general Hard decision upper bound for a Binary Symmetric Channel (BSC) is given by [2]:

$$P_e \leq \frac{1}{\delta} \sum_{d^{(\beta)}=d_f}^{\infty} c_{d^{(\beta)}} P_{d^{(\beta)}}. \quad (1)$$

  where δ is the puncturing period, d_f is the free distance of the code, and $c_{d^{(\beta)}}$ is the coefficient of the bit input weight enumeration function (IWEF) of a given code β.

- In this study we derive an expression for $P_{d^{(\beta)}}$ for the bits in class i as given by the following theorem.
- **Theorem:** $\forall d^{\beta} \in \mathbb{N}$, $P_{d^{(\beta)}}$ is given by

$$P_{d^{(\beta)}} = \sum_{k=\frac{d^{(\beta)}+1+[d^{(\beta)}+1]_2}{2}}^{d^{(\beta)}} \binom{d^{(\beta)}}{k} (\epsilon_0)^k (1-\epsilon_0)^{d^{(\beta)}-k} + \frac{[d^{(\beta)}+1]_2}{2} \binom{d^{(\beta)}}{\frac{d^{(\beta)}}{2}} (\epsilon_0)^{\frac{d^{(\beta)}}{2}} (1-\epsilon_0)^{\frac{d^{(\beta)}}{2}}$$

  where $[.]_2$ is modulo two equivalent of the argument and $\epsilon_0 = \epsilon_1 - p_1(\epsilon_1 - \epsilon_2)$. Finally, we note a similar argument can be given for the LP bits.

- **Corollary:** In general for $\epsilon_1, \ldots, \epsilon_L$, we have $\epsilon_0 = \epsilon_1 - \sum_{i=2}^{L}(\epsilon_1 - \epsilon_i)p_{L-i+1}$

[3] *J. Hagenauer, Rate-Compatible Punctured Convolutional Codes (RCPC Codes) and Their Applications, IEEE Trans. on Commun., vol. 36, No. 4, pp. 389-400, April 1997.*

Sketch of the Proof (1)

- Assume all-zero binary sequency is transmitted. Consider the following figure:

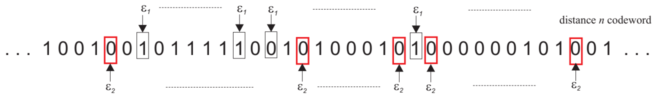

Figure: HP bit stream going through a channel with varying bit error probability.

Thus, the average probability of selecting the wrong path in trellis $\mathcal{T}$ is given by

$$P_n = \sum_{s=0}^{n} Pr\{s; n\}\psi_s(k \geq \frac{n+1}{2}) \quad (3)$$

where $Pr\{s; n\}$ is the probability of having exactly s positions that have a BER ϵ_1, and $\psi_s(k \geq \frac{n+1}{2})$ is the probability of selecting an incorrect path when exactly s positions have the BER ϵ_1.

- **lemma:** For n odd, P_n is given by:

$$P_n = \sum_{k=\frac{n+1}{2}}^{n} \sum_{m=0}^{k} \sum_{s=m}^{n-k+m} (1-p)^s p^{n-s} \binom{n}{s} \binom{s}{m} \epsilon_1^m (1-\epsilon_1)^{s-m} \binom{n-s}{k-m} \epsilon_2^{k-m} (1-\epsilon_2)^{n-s-k+m} \quad (1)$$

where $\binom{a}{b} = 0$ if $b > a$.

- *Proof:* Please see the paper.

Sketch of the Proof (2)

- **Proposition :** $\forall \mathfrak{p} \in \mathbb{N}^{+}$, $n = 2\mathfrak{p} + 1$, we have $P_n = \widetilde{P}_n$ where

$$\widetilde{P}_n = \sum_{k=\frac{n+1}{2}}^{n} \binom{n}{k} \left((1-p)\epsilon_1 + p\epsilon_2\right)^k \left(1-(1-p)\epsilon_1 - p\epsilon_2\right)^{n-k} \tag{2}$$

- *Proof:* Please see the paper. This completes the proof of the **Theorem** for n odd because $\epsilon_0 = (1-p)\epsilon_1 + p\epsilon_2 = \epsilon_1 - p(\epsilon_1 - \epsilon_2)$.
- If n is even, we have $\mathrm{k} = n/2 + 1$. An incorrect path is chosen when the number of errors exceeds $n/2$ If it equals $n/2$, the decoder selects one of the paths randomly. Thus,

$$P_n = \sum_{s=0}^{n} Pr\{s; n\}\psi_s(k \geq \frac{n}{2} + 1) + \frac{1}{2}\mathfrak{P}(n/2) \tag{3}$$

where the $1/2$ comes from the fact that half of the time the decoder incurs an error and $\mathfrak{P}(n/2)$ is the probability of selecting an incorrect path when the number of errors equals $n/2$. We can show that

$$\mathfrak{P}(n/2) = \binom{n}{n/2} \left((1-p)\epsilon_1 + p\epsilon_2\right)^{n/2} \left(1-(1-p)\epsilon_1 - p\epsilon_2\right)^{n/2} \tag{4}$$

which completes the proof of the **Theorem** for n even.

Simulation parameters

- Throughout this section, we use two types of codes: (1) RCPC code with memory M=6 and M=4, (2) NASA standard code: (7, 1/2) convolutional code.

Code	Rate	N	GP	δ	c_d
CC	1/2	6	[171, 131]	1	$c_{10} = 36, c_{11} = 0, \ldots$
RCPC	8/9	6	[133, 171, 65]	8	$c_3 = 24, c_4 = 740, \ldots$
RCPC	2/3	6	[133, 171, 65]	8	$c_6 = 12, c_7 = 280, \ldots$
RCPC	1/2	4	[13, 29, 17, 27]	8	$c_7 = 32, c_8 = 96, \ldots$
RCPC	2/3	4	[13, 29, 17, 27]	8	$c_4 = 4, c_5 = 0, \ldots$
RCPC	2/5	4	[13, 29, 17, 27]	8	$c_8 = 2, c_9 = 34, \ldots$

Table: Parameters of the codes. GP: Generator polynomial.

- We tested both AWGN and slowly varying flat Rayleigh fading channels using various hierarchical constellations: H- 4PAM, H-16QAM, two concentric hierarchical 2/4 PSK [3].
- For the given simulation parameters, we use the triple $(\epsilon_1, \epsilon_2, p_1)$ to calculate the upper bound.

[4] P.K. Vitthaladevuni and M.S. Alouini, Exact BER Computation of Generalized Hierarchical PSK Constellations,

IEEE Trans. on Commun., vol. 51, No. 12, pp. 2030-2037, Dec. 2003.

Simulation results (1)

- Simulation results for H-4PAM with $\alpha = 3$ and H-16QAM with varying α values are shown below. Note that inphase and quadrature components of a given H-16QAM can be thought of as two independent H-4PAM constellations. Only few terms in the sum (1) are used to calculate the bounds.

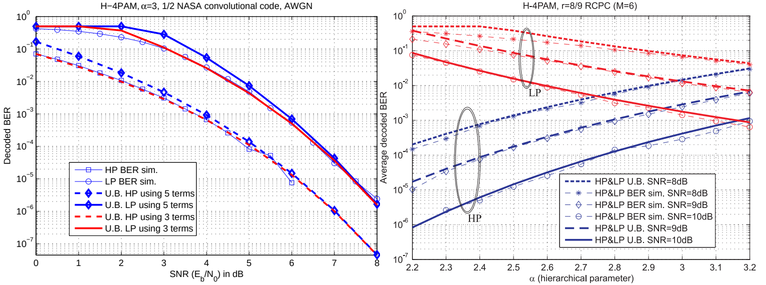

Figure: Simulation results under AWGN.

Simulation results (2)

- We consider the HP bit location of two concentric 2/4 PSK constellation and choose $\theta_2 = \pi/12$ and $\varsigma_2 = 2 \times \varsigma_1$. Thus, we have $\theta_1 = \arccos\left(2\sin\frac{\pi}{12}\right)$.

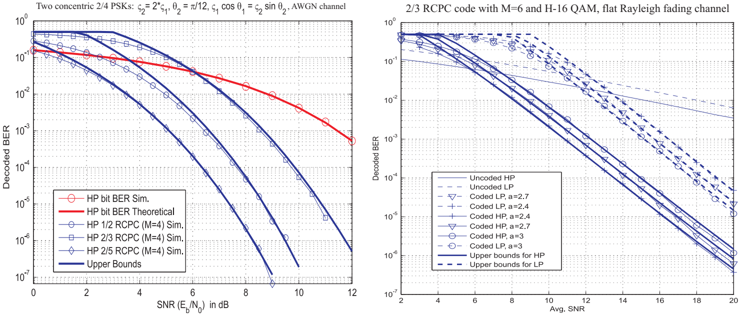

Figure: Simulation results under AWGN and Flat Rayleigh channels.

- The second plot shows a Monte Carlo simulation result using a 2/3 RCPC code with M=6 and H-16QAM as well as the bounds.

Conclusions & References

CONCLUSIONS:

- We extended the hard decision upper bound results for BSC to coded hierarchical modulations.
- The proposed bounds are efficient and easier to calculate than previous studies.
- Bounds show a good approximation to the numerical results for the coded hierarchical modulation under both AWGN and Flat Rayleigh fading channels.
- The results can be extended to channels with memory as long as we are able to compute the error probabilities $\{\epsilon_i\}_{i=1}^{L}$ and $\{p_i\}_{i=1}^{L}$.

REFERENCES:

(1) *P.K. Vitthaladevuni and M.S. Alouini, An upper bound on the BER of block coded hierarchical constellations, 2003 IEEE Paci?c Rim Conference on. Communications, Computers and Signal Processing (PACRIM 03), vol. 2, pp. 950-953. Aug. 2003.*

(2) *J.H. Lim and S.B. Gelfand, Performance analysis of hierarchical coded modulation systems, Global Telecommunications Conference, 2000*

(3) *A.J. Viterbi, Convolutional codes and their performance in communication systems, IEEE Trans. on Commun. Technol. vol. COM-19, pp. 751-771, 1971.*